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Ultrafilters, monotone functions and pseudocompactness

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Abstract. In this article we, given a free ultrafilter p on ω, consider the following classes of ultrafilters:

(1) $T(p)$ - the set of ultrafilters Rudin-Keisler equivalent to p,
(2) $S(p) = \{q \in \omega^* : \exists\ f \in \omega^\omega,$ strictly increasing, such that $q = f^\beta(p)\}$,
(3) $I(p)$ - the set of strong Rudin-Blass predecessors of p,
(4) $R(p)$ - the set of ultrafilters equivalent to p in the strong Rudin-Blass order,
(5) $P_{RB}(p)$ - the set of Rudin-Blass predecessors of p, and
(6) $P_{RK}(p)$ - the set of Rudin-Keisler predecessors of p,

and analyze relationships between them. We introduce the semi-P-points as those ultrafilters $p \in \omega^*$ for which $P_{RB}(p) = P_{RK}(p)$, and investigate their relations with P-points, weak-P-points and Q-points. In particular, we prove that for every semi-P-point p its α-th left power ${}^\alpha p$ is a semi-P-point, and we prove that non-semi-P-points exist in ZFC. Further, we define an order $\trianglelefteq$ in $T(p)$ by $r \trianglelefteq q$ if and only if $r \in S(q)$. We prove that $(S(p), \trianglelefteq)$ is always downwards directed, $(R(p), \trianglelefteq)$ is always downwards and upwards directed, and $(T(p), \trianglelefteq)$ is linear if and only if p is selective.

We also characterize rapid ultrafilters as those ultrafilters $p \in \omega^*$ for which $R(p) \setminus S(p)$ is a dense subset of ω^*.

A space X is M-pseudocompact (for $M \subset \omega^*$) if for every sequence $(U_n)_{n<\omega}$ of disjoint open subsets of X, there are $q \in M$ and $x \in X$ such that $x = q\text{-}\lim(U_n)$; that is, $\{n < \omega : V \cap U_n \neq \emptyset\} \in q$ for every neighborhood V of x. The $P_{RK}(p)$-pseudocompact spaces were studied in [ST].

In this article we analyze M-pseudocompactness when M is one of the classes $S(p)$, $R(p)$, $T(p)$, $I(p)$, $P_{RB}(p)$ and $P_{RK}(p)$. We prove that every Frolík space is $S(p)$-pseudocompact for every $p \in \omega^*$, and determine when a subspace $X \subset \beta\omega$ with $\omega \subset X$ is M-pseudocompact.

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0. Introduction, basic definitions and preliminaries

As usual, ω represents the set of finite ordinals with the discrete topology. Its Stone-Čech compactification, $\beta\omega$, is considered as the set of ultrafilters on ω equipped with the Stone topology. The remainder $\omega^* = \beta\omega \setminus \omega$ consists of the free ultrafilters on ω. For a function $f : \omega \to \omega$, $f^\beta : \beta\omega \to \beta\omega$ denotes the continuous extension of f. Note that $f^\beta(p) \in \omega^*$ as long as f is not constant on a set in p. The *Rudin-Keisler* (pre)-order on ultrafilters is defined by $p \leq_{RK} q$ if there is an $f \in \omega^\omega$ such that $p = f^\beta(q)$.

Two ultrafilters p, q are of the same type ($p \approx q$) if there is a permutation of ω sending one ultrafilter to the other. For a subset B of ω, a function $f : B \to \omega$ is *strictly increasing* (resp., *non-decreasing*) on $A \subset B$ if $f(n) < f(m)$ (resp., $f(n) \leq f(m)$) for each $n < m \in A$; and $f : B \to \omega$ is *finite-to-one* on $A \subset B$ if $|f^{-1}(k) \cap A| < \omega$ for each $k \in \omega$. A function f is *strictly increasing* (resp., *non-decreasing*, *finite-to-one*) if it is *strictly increasing* (resp., *non-decreasing*, *finite-to-one*) on its domain. As usual, if $f : X \to Y$ is a function and $A \subset X$, then $f \restriction A$ denotes the restriction of f to A. By $Sym(\omega)$ we denote the set of permutations on ω and by ${}^{\omega\nearrow}\omega$ the set of strictly increasing functions. $Nd(\omega)$ denotes the set of non-decreasing functions from ω to ω, and $Fo(\omega)$ consists of the finite-to-one functions from ω to ω. If A is a subset of ω, ${}^{A\nearrow}\omega = \{f \in \omega^\omega : f$ is strictly increasing on $A\}$. Finally, $P({}^{\omega\nearrow}\omega)$ consists of functions defined on an infinite subset of ω which are strictly increasing in their domains.

The *Rudin-Blass* order, $\leq_{RB}$, is the variant of the Rudin-Keisler order where the witnessing function is required to be finite-to-one and *strong Rudin-Blass* order, $\leq_{RB^+}$ requires the witnessing function to be non-decreasing. For more information on these orderings consult [vM] and [LZ].

Definition 0.1. *Let $p \in \omega^*$. Let:*

(1) $T(p) = \{q \in \omega^* : \exists\ \sigma \in Sym(\omega) \quad q = \sigma^\beta(p)\}$,
(2) $S(p) = \{q \in \omega^* : \exists\ f \in \omega^{\nearrow\omega} \quad q = f^\beta(p)\}$,
(3) $R(p) = \{q \in \omega^* : \exists\ A \in p,\ f \in {}^{A\nearrow}\omega \quad q = f^\beta(p)\}$,
(4) $I(p) = \{q \in \omega^* : \exists\ f \in \omega^\omega$ non-decreasing $\quad q = f^\beta(p)\}$,
(5) $P_{RB}(p) = \{q \in \omega^* : \exists\ f \in \omega^\omega$ finite-to-one $\quad q = f^\beta(p)\}$,
(6) $P_{RK}(p) = \{q \in \omega^* : \exists\ f \in \omega^\omega \quad q = f^\beta(p)\}$.

In Section 1 we study the relationship between the aforementioned classes in connection with special properties of ultrafilters. Note that the last two classes correspond to Rudin-Blass and Rudin-Keisler predecessors of p, while the class $I(p)$ is (in the terminology of [LZ]) the class of strong Rudin-Blass predecessors of p. In [LZ] it is shown, among other things, that $P_{RB}(p) = \bigcup\{T(q) : q \in I(p)\}$. The set $R(p)$ is the strong Rudin-Blass equivalence class of p, as we will verify later.

In Section 2, we introduce the notion and investigate the properties of *semi-P-points*, ultrafilters for which the classes of Rudin-Keisler and Rudin-Blass predecessors coincide. In particular we prove that for a semi-P-point p, its left α-th power ${}^\alpha p$ is still a semi-P-point, and we prove that non-semi-P-points exist in ZFC.

It will also be shown, that this finer classification of ultrafilters, produces (for every $p \in \omega^*$) an ordering $\trianglelefteq$ (not a pre-ordering!) on the type of p (i.e. the class of ultrafilters indistinguishable from p), defined by $s \trianglelefteq q \Leftrightarrow s \in S(q)$ (see Definition 3.1 below). Properties of this order will be studied in Section 3.

The structure of ultrafilters on a countable set has been extensively studied during the past decades by both set-theorists and by topologists. They used purely set-theoretic methods to get topological concepts and to study topological properties of $\beta\omega$ and ω^* on one hand and on the other hand, translating topological facts about subspaces of $\beta\omega$ to obtain information about the partial preorders defined on $\beta\omega$ or ω^*.

An example of this is the development of the study of covering and convergence properties of spaces modulo ultrafilters. An instance of this is the study of p-pseudocompactness and $P_{RK}(p)$-pseudocompactness introduced and analyzed in [GF] and [ST], respectively.

In this article we are going to refine the concept of $P_{RK}(p)$-pseudocompactness by considering a finer classification of ultrafilters based on the image via a strictly increasing function: $S(p)$-pseudocompactness, $R(p)$-pseudocompactness, etc. In Section 4 we will prove that every pseudocompact space belonging to the Frolík class satisfies these new conditions, and in Section 5 we will analyze those subspaces of $\beta\omega$ which contain ω and possess these new properties.

The following are (with one exception) standard:

Definition 0.2. *Let $p \in \omega^*$. Then:*

(1) p is a P-point if for every partition $\{I_n : n \in \omega\}$ of ω into sets not in p there is an $A \in p$ such that $|A \cap I_n| < \aleph_0$ for every $n \in \omega$,
(2) p is a Q-point if for every partition $\{I_n : n \in \omega\}$ of ω into finite sets there is an $A \in p$ such that $|A \cap I_n| \leq 1$ for every $n \in \omega$,
(3) p is a Q'-point if for every partition $\{I_n : n \in \omega\}$ of ω into intervals there is an $A \in p$ such that $|A \cap I_n| \leq 1$ for every $n \in \omega$,
(4) p is selective if for every partition $\{I_n : n \in \omega\}$ of ω into sets not in p there is an $A \in p$ such that $|A \cap I_n| \leq 1$ for every $n \in \omega$.
(5) p is rapid if for each function $h \in \omega^\omega$, there is $A \in p$ with $|A \cap h(n)| \leq n$ for every $n < \omega$.

It is obvious that every selective ultrafilter is both a P-point and a Q-point, and that every Q-point is a Q'-point. Also, p is selective if and only if it is both a P-point and a Q-point. Another property of selective ultrafilter p needed later on in the text is its Ramsey property: For every coloring $\phi : [\omega]^2 \longrightarrow 2$ there is an $A \in p$ such that $|\phi[[A]^2]| = 1$; in other words, all pairs in A are colored by the same color. The notion of a Q'-point is merely a matter of convenience and it will be shown in the text that p is a Q-point if and only if p is a Q'-point. Another way to say that p is rapid is: For every sequence $d_0 < d_1 < \ldots < d_n < \ldots$ there is an $A \in p$ such that $d_i < a_i$ for every i, where $A = \{a_i : i < \omega\}$ and $a_i < a_{i+1}$ for all $i < \omega$. It is easily seen that every Q-point is rapid.

The following well known facts will be used very often in the text.

Lemma 0.3. *Let $f \in \omega^\omega$ and $p \in \omega^*$. Then*

(1) $f^\beta(p) = \{A \subset \omega : f^{-1}(A) \in p\}$.
(2) $f^\beta(p) = p$ if and only if $\{n \in \omega : f(n) = n\} \in p$.
(3) $f^\beta(p) \approx p$ if and only if there is $A \in p$ such that $f \restriction A$ is one-to-one.

Lemma 0.4. *Let $f, g \in \omega^\omega$ and $p \in \omega^*$.*

(1) If, for an $A \in p$, $f \restriction A = g \restriction A$, then $f^\beta(p) = g^\beta(p)$.
(2) (Z. Frolík) If f or g is one-to-one, then $f^\beta(p) = g^\beta(p)$ if and only if $E_{f,g} = \{n \in \omega : f(n) = g(n)\} \in p$.

Note that clause (2) of Lemma 0.4 does not necessarily hold for arbitrary f and g (see Example 2.3).

A trivial fact we are also going to use frequently is:

Lemma 0.5. *Let $f \in \omega^{\nearrow\omega}$. Then $f(n + k) \geq f(n) + k$ for every $n, k \in \omega$. In particular, $f(n) \geq n$ for every $n \in \omega$.*

Given a space X, a point $p \in \omega^*$ and a sequence $s = (F_n)_{n<\omega}$ of subsets of X, we say that a point $x \in X$ is a *p-limit of s* (in symbols, $x = p\text{-lim}(F_n)$) if $\{n < \omega : F_n \cap V \neq \emptyset\} \in p$ for each neighborhood V of x (see [GS]).

Let $\mathcal{C}$ be a collection of subsets of a space X, and M be a subset of ω^*. We say that X is *$M_\mathcal{C}$-compact* if for every sequence $(F_n)_{n<\omega}$ of disjoint elements of $\mathcal{C}$, there are $x \in X$ and $p \in M$ such that $x = p\text{-lim}(F_n)$. If $\mathcal{C}$ is the collection of singletons of X, then $M_\mathcal{C}$-compactness coincides with M-compactness (see [B]), and when $\mathcal{C}$ is the collection of open subsets of X, then $M_\mathcal{C}$-compactness is called M-pseudocompactness.

Observe that every compact space X is $M_\mathcal{C}$-compact for every collection $\mathcal{C}$ of subsets of X and every $M \subset \omega^*$. Moreover, if $M \subset N \subset \omega^*$ and $\mathcal{D}$ refines $\mathcal{C}$, then every $M_\mathcal{D}$-compact space is $N_\mathcal{C}$-compact. In particular, N-compactness implies N-pseudocompactness. Also, if X is M-compact, then X is countably compact, and if X is M-pseudocompact, then X is pseudocompact.

We will focus our attention on analyzing M-pseudocompactness, when M is one of the sets considered in Definition 0.1. Recall the following standard fact.

Lemma 0.6. *Let $r, p \in \omega^*$ and $f \in \omega^\omega$. Then $r = p\text{-lim}(f(n))$ if and only if $f^\beta(p) = r$.*

A slight generalization of Lemma 0.6 yields:

Proposition 0.7. *Let $p \in \omega^*$ and $f : X \to Y$ be a continuous function. Let $(F_n)_{n<\omega}$ be a sequence of subsets of X. If $x = p\text{-lim}(F_n)$, then $f(x) = p\text{-lim}(f(F_n))$.*

Corollary 0.8. *For every $M \subset \omega^*$, M-compactness and M-pseudocompactness are preserved by continuous functions.*

Moreover, for every $M \subset \omega^*$, M-compactness is hereditary with respect to closed subsets, and M-pseudocompactness is inherited by regular closed subsets.

Using Lemma 0.3, it is easy to prove the following (for a proof see [ST, Lemma 2.1.(1)]).

Lemma 0.9. *Let $p \in \omega^*$, $f : \omega \to \omega$ and let $(F_n)_{n<\omega}$ be a sequence of subsets of X. Then, for an $x \in X$,*

$$x = f^\beta(p)\text{-}\lim(F_n) \Leftrightarrow x = p\text{-}\lim(F_{f(n)}).$$

Convention 0.10. Throughout the paper we will use the following convention: If a capital letter, say A, denotes an infinite subset of ω, then the lower case letters $a_0, a_1, \ldots, a_n, \ldots$ denote its elements in an increasing way ($a_i < a_{i+1}$). Moreover, the lower case letter a denotes the strictly increasing function which lists the elements of A; that is $a(n) = a_n$.

The proofs of the following assertions are standard.

Lemma 0.11. *Let $f \in \omega^\omega$ and $A \subset \omega$.*

(1) If $f \restriction A$ is a non-decreasing function, then there exists $g \in Nd(\omega)$ such that $g \restriction A = f \restriction A$.
(2) If $f \restriction A$ is a finite-to-one function, then there exists $g \in Fo(\omega)$ such that $g \restriction A = f \restriction A$.
(3) There is $g \in {}^{\omega\nearrow}\omega$ such that $g \restriction A = f \restriction A$ if and only if $f \restriction A$ is strictly increasing, $a_0 \leq f(a_0)$ and $a_{n+1} - a_n \leq f(a_{n+1}) - f(a_n)$ for every $n < \omega$.

Lemma 0.4 and Lemma 0.11 produce:

Corollary 0.12. *Let $f \in \omega^\omega$ and $p \in \omega^*$.*

(1) If $q = f^\beta(p)$ and $f \restriction A$ is a non-decreasing function for an $A \in p$, then there exists $g \in Nd(\omega)$ such that $g^\beta(p) = q$.
(2) If $q = f^\beta(p)$ and $f \restriction A$ is a finite-to-one function for an $A \in p$, then there exists $g \in Fo(\omega)$ such that $g^\beta(p) = q$.
(3) There is $g \in {}^{\omega\nearrow}\omega$ such that $f^\beta(p) = g^\beta(p)$ if and only if there is $A \in p$ such that $f \restriction A$ is strictly increasing, $a_0 \leq f(a_0)$ and $a_{n+1} - a_n \leq f(a_{n+1}) - f(a_{n+1})$ for every $n < \omega$.
(4) We can always find $A \in p$ and a function $g : \omega \to \omega$ such that: (a) $g \restriction A = f \restriction A$, (b) $g(\omega \setminus A) \subset \omega \setminus g(A)$, and (c) $g^\beta(p) = f^\beta(p)$.

1. Relationships between the classes $T(p)$, $S(p)$, $R(p)$, $I(p)$, $P_{RB}(p)$ and $P_{RK}(p)$

As a consequence of Corollary 0.12, we obtain that $I(p) = \{q \in \omega^* : \exists\ A \in p$ and $f \in \omega^\omega$ such that f is non-decreasing on A and $q = f^\beta(p)\}$ and $P_{RB}(p) = \{q \in \omega^* : \exists\ A \in p$ and $f \in \omega^\omega$ such that f is finite-to-one on A and $q = f^\beta(p)\}$.

Of course, if f and g are strictly increasing (resp. non-decreasing, finite-to-one) and the range of g is contained in the domain of f, then $f \circ g$ is strictly increasing (resp., non-decreasing, finite-to-one). Moreover, every strictly increasing function is a non-decreasing function, every non-decreasing function is finite-to-one, and if $f : \omega \to \omega$ is strictly increasing, then $f^{-1} : f[\omega] \to \omega$ is strictly increasing too.

Proposition 1.1. *Let $p \in \omega^*$. Then:*

(1) $p \in S(p) \subset R(p) \subset T(p) \subset P_{RB}(p) \subset P_{RK}(p)$,
(2) $R(p) \subset I(p) \subset P_{RB}(p)$,
(3) $R(p) = T(p) \cap I(p)$.

Proof. The only part not entirely trivial in (1) and (2) is $R(p) \subset T(p)$. To see this, let $f \in \omega^\omega$ be strictly increasing on $A \in p$. We can assume, without loss of generality, that $|\omega \setminus A| = \aleph_0$. Extend $f \restriction A$ to a permutation σ. Then by Lemma 0.4, $\sigma^\beta(p) = f^\beta(p)$.

In order to prove (3), take $r \in T(p) \cap I(p)$. There exist $f \in Sym(\omega)$ and $g \in Nd(\omega)$ such that $f^\beta(p) = r = g^\beta(p)$. Let A be the set $\{n < \omega : f(n) = g(n)\}$. Then $A \in p$ and f (and g) is strictly increasing on A. Therefore, $r \in R(p)$. □

We can summarize elementary relationships between the classes as follows:

Theorem 1.2. *Let $r, p \in \omega^*$. Then*

(1) $S(p) \neq R(p)$; in particular, $S(p) \neq T(p)$ and $S(p) \neq I(p)$,
(2) $r \in S(p)$ if and only if $S(r) \subset S(p)$,
(3) $S(r) = S(p)$ if and only if $r = p$,
(4) $r \in I(p)$ if and only if $I(r) \subset I(p)$,
(5) $r \in R(p)$ if and only if $R(r) \subset R(p)$,
(6) $r \in R(p)$ if and only if $R(r) = R(p)$,
(7) $R(r) = R(p)$ or $R(r) \cap R(p) = \emptyset$,
(8) $S(r) \cap S(p) \neq \emptyset$ if and only if $R(r) = R(p)$.
(9) $r \notin I(p)$ if and only if $R(r) \subset \omega^* \setminus I(p)$, if and only if $R(r) \cap (\omega^* \setminus I(p)) \neq \emptyset$.

Proof. (1) Let $A \in p$ such that $0 \notin A$ and $|\omega \setminus A| = \aleph_0$. Define $g : \omega \to \omega$ by $g(n) = n - 1$ if $n \in A$, and $g(n) = 0$ otherwise. Then, $r = g^\beta(p) \in R(p)$. If for some $f \in {}^{\omega\nearrow}\omega$, $f^\beta(p) = r$, then $\{n < \omega : f(n) = g(n)\} \in p$, hence there is an $n \in \omega$ for which $f(n) < n$, but this is not possible (Lemma 0.5). Therefore, $r \notin S(p)$.

(2) and (4) are easy to prove.

(3) The reverse implication is trivial. We prove the direct implication. Assuming $S(r) = S(p)$, we have functions $f, g \in {}^{\omega\nearrow}\omega$ such that $r = f^\beta(p)$ and $p = g^\beta(r)$. Since $(f \circ g)^\beta(r) = r$, $f \circ g$ is the identity function on a set $B \in r$. On B, both f and g must be the identity because of monotonicity. So $p = g^\beta(r) = r$.

(5) Since $r \in R(p)$, there is a function $f : \omega \to \omega$ and there is $A \in p$ such that $f \restriction A$ is strictly increasing, and $f^\beta(p) = r$. If $s \in R(r)$, we can find $g : \omega \to \omega$ such that $g^\beta(r) = s$ and a $B \in r$ on which g is strictly increasing. Since $B \in r$, $f^{-1}(B) \in p$. Let $C = A \cap f^{-1}(B)$. Then $(g \circ f) \restriction C$ is strictly increasing, $C \in p$ and $(g \circ f)^\beta(p) = s$. So, $s \in R(p)$.

(6) Let $r \in R(p)$. By (5), $R(r) \subset R(p)$. Moreover, there are $f : \omega \to \omega$ and $A \in p$ such that $f \restriction A$ is strictly increasing and $f^\beta(p) = r$. We have then that $f[A] \in r$ and $f^{-1} : f[A] \to \omega$ is strictly increasing. Let $h : \omega \to \omega$ be defined by $h(n) = f^{-1}(n)$ if $n \in f[A]$, and $h(n) = 0$ if $n \notin f[A]$. Then $\{n < \omega : (h \circ f)(n) = n\} \in p$. So, $h^\beta(r) = h^\beta(f^\beta(p)) = (h \circ f)^\beta(p) = p$. But h is strictly increasing in an element of r, so $p \in R(r)$, and this means that $R(p) \subset R(r)$.

(7) This is a consequence of (6).

(8) If $S(r) \cap S(p) \neq \emptyset$ then $R(r) \cap R(p) \neq \emptyset$. Using (6) we get $R(r) = R(p)$.

Now, assume that $R(r) = R(p)$. Let $A \in p$ and $h \in {}^{A\nearrow}\omega$ be such that $h^\beta(p) = r$. By Corollary 0.12.(4) we can assume that $h(\omega \setminus A) \subset \omega \setminus h(A)$. Let $b_n = h(a_n)$ and $B = \{b_n : n < \omega\}$. Note that $b_n < b_m$ if $n < m$. Define $\psi : B \to \omega$ by $\psi(b_n) = b_0 + ... + b_n + a_0 + ... + a_n$. By Lemma 0.11.(3) there are two strictly increasing functions $f, g \in {}^{\omega\nearrow}\omega$ which extend ψ and $h' = \psi \circ h \upharpoonright A$, respectively.

Claim. $g^\beta(p) = f^\beta(r)$.

In fact, by definition of f and g, and using that the domain of ψ is $B = h[A]$, we have that the composite $f \circ h$ agrees with g on the set $A \in p$. So, $f^\beta(r) = f^\beta(h^\beta(p)) = g^\beta(p)$.

(9) If $R(r) \subset \omega^* \setminus I(p)$, then $r \notin I(p)$ because $r \in R(r)$. Now, assume that $q \in R(r) \cap I(p)$. Then, there exists $f \in {}^{A\nearrow}\omega$ with $A \in q$ such that $f^\beta(q) = r$ (because of (6)), and there is $g \in Nd(\omega)$ such that $g^\beta(p) = q$. Then, $(f \circ g)^\beta(p) = f^\beta(g^\beta(p)) = f^\beta(q) = r$. Then $A \in q$, so $g^{-1}(A) \in p$ and $(f \circ g) \upharpoonright g^{-1}(A)$ is non-decreasing. So $r \in I(p)$. □

The well-known minimality of selective ultrafilters (Q-points) in the Rudin-Keisler order (Rudin-Blass order) translates directly into:

Lemma 1.3. *Let $p \in \omega^*$. Then:*

(1) p is a Q-point if and only if $T(p) = P_{RB}(p)$,
(2) p is selective if and only if $T(p) = P_{RK}(p)$.

Lemma 1.4. *$p \in \omega^*$ is a Q'-point if and only if $I(p) \subseteq T(p)$.*

Proof. For the direct implication assume that p is a Q'-point and let $q \in I(p)$. Then there is an $h \in \omega^\omega$ non-decreasing such that $q = h^\beta(p)$. Let $I_n = h^{-1}(\{n\})$. The family $\{I_n : n \in \omega\}$ constitutes a partition of ω into intervals, so there is an $A \in p$ ($|\omega \setminus A| = \aleph_0$) such that $|A \cap I_n| \leq 1$ for every $n \in \omega$. The function $h \upharpoonright A$ is then strictly increasing. Extend $h \upharpoonright A$ to a permutation σ. By Lemma 0.4, $q = h^\beta(p) = \sigma^\beta(p)$, and hence $q \in T(p)$.

For the reverse implication let $\{I_n : n \in \omega\}$ be an increasing enumeration of a partition of ω into intervals and let $f(m) = n$ if and only if $m \in I_n$. As $I(p) \subseteq T(p)$, there is a permutation σ such that $f^\beta(p) = \sigma^\beta(p)$ and by Lemma 0.4, $E_{f,\sigma} \in p$. As f is constant on each I_n, $|E_{f,\sigma} \cap I_n| \leq 1$. So, p is a Q'-point. □

Lemma 1.5. *$p \in \omega^*$ is a Q-point $\Leftrightarrow$ p is a Q'-point $\Leftrightarrow$ $T(p) \subseteq I(p)$.*

Proof. Let p be a Q'-point, $\sigma \in Sym(\omega)$ and $q = \sigma^\beta(p)$.

Case 1. There is a strictly increasing sequence $\{n_i : i \in \omega\}$ such that $n_0 = 0$ and $\sigma[[n_i, n_{i+1})] = [n_i, n_{i+1})$ for every $i \in \omega$.

Let $I_i = [n_i, n_{i+1})$. As p is a Q'-point there is an $A \in p$ such that $|A \cap I_i| \leq 1$ for every $i \in \omega$.

Case 2. The set $\{k \in \omega : \sigma[[0,k)] = [0,k)\}$ is bounded.

Construct two sequences $\{n_i : i \in \omega\}$ and $\{m_i : i \in \omega\}$ of integers by putting

(1) $n_0 = m_0 = \max\{k \in \omega : \sigma[[0,k)] = [0,k)\}$
(2) $m_1 = \sigma(n_0)$,
(3) $n_{i+1} = \max \sigma^{-1}[[m_i, m_{i+1})] + 1$ and
(4) $m_{i+1} = \max \sigma[[n_i, n_{i+1})] + 1$.

Note that $\sigma[[n_i, n_{i+1})] \subseteq [m_i, m_{i+2})$ for every $i \in \omega$. Let

$$I = \bigcup_{i\in\omega}([n_i, n_{i+1}) \cap \sigma^{-1}[[m_{i+1}, m_{i+2})])$$

and

$$J = \bigcup_{i\in\omega}([n_i, n_{i+1}) \cap \sigma^{-1}[[m_i, m_{i+1})]).$$

Then exactly one of I and J is in p, say I (the case for J is analogous). Let $I_i = [n_i, n_{i+1})$. As p is a Q'-point there is a $B \in p$ such that $|B \cap I_i| \leq 1$ for every $i \in \omega$. Let $A = B \cap I$.

In both cases $\sigma \restriction A$ is an increasing function. Let $f \in \omega^\omega$ be a non-decreasing extension of $\sigma \restriction A$. Again by Lemma 0.4, $f^\beta = \sigma^\beta$ hence $T(p) \subseteq I(p)$.

Now assume that $T(p) \subseteq I(p)$ and let $\{I_n : n \in \omega\}$ be any partition of ω into finite sets. Let σ be a permutation of ω such that $\sigma \restriction I_n$ is (strictly) decreasing for every $n \in \omega$. As $T(p) \subseteq I(p)$, there is a non-decreasing f such that $f^\beta = \sigma^\beta$. By Lemma 0.4, $E_{f,\sigma} \in p$. As f is decreasing on each I_n, $|E_{f,\sigma} \cap I_n| \leq 1$ and so p is a Q-point.

To close the circle of implications it is enough to note that every Q-point is trivially a Q'-point. □

Note that an easy modification of the proof yields:

Lemma 1.6. *p is a Q-point if and only if for every finite-to-one function f there is an $A \in p$ such that $f \restriction A$ is strictly increasing.*

Now we can summarize the results in the following theorem:

Theorem 1.7. *Let $p \in \omega^*$. Then:*

(1) The following are equivalent:
(a) p is a Q-point
(b) p is a Q'-point
(c) $I(p) \subseteq T(p)$
(d) $T(p) \subseteq I(p)$
(e) $I(p) = T(p)$
(f) $T(p) = P_{RB}(p)$
(g) $I(p) = P_{RB}(p)$.
(h) $R(p) = P_{RB}(p)$.

(2) The following are equivalent:
(a) p is selective
(b) $T(p) = P_{RK}(p)$
(c) $I(p) = P_{RK}(p)$.
(d) $R(p) = P_{RK}(p)$.
(3) If p is a P-point then $P_{RB}(p) = P_{RK}(p)$.

Proof. Clause (1) follows easily from Lemma 1.3, Lemma 1.4, Lemma 1.5 and 1.1.(3). Clause (2) follows from Lemma 1.3 and clause (1) using the fact that every selective ultrafilter is a Q-point. So the only thing that requires argumentation is clause (3). Let p be a P-point and let $q \in P_{RK}(p)$. Then there is an $f \in \omega^\omega$ such that $q = f^\beta(p)$. Let $I_n = f^{-1}(\{n\})$. Then $\{I_n : n \in \omega\}$ is a partition of ω and each $I_n \notin p$ (as otherwise $q \notin \omega^*$). As p is a P-point there is an $A \in p$ such that $A \cap I_n$ is finite for every $n \in \omega$. Extend $f \restriction A$ to any finite-to-one function g. Then $q = f^\beta(p) = g^\beta(p)$ and hence $q \in P_{RB}(p)$. □

It is convenient to introduce the following definition:

Definition 1.8. *Call an ultrafilter $p \in \omega^*$ a semi-P-point if $P_{RB}(p) = P_{RK}(p)$.*

It is known that there are (in ZFC!) points which are not semi-P-points (see [vM] or Section 4). On the other hand, the existence of P-points, Q-points and selective ultrafilters is not provable in ZFC alone (see, for example, [BJ]). Note that if p is a Q-point which is not selective, then $P_{RB}(p) \neq P_{RK}(p)$. For every free ultrafilter p there are only four possible scenarios:

(1) All classes mentioned above are distinct.
(2) $S(p) \neq R(p) = T(p) = I(p) = P_{RB}(p) \neq P_{RK}(p)$, which happens exactly when p is a Q-point and not a (semi-)P-point,
(3) $S(p) \neq R(p) = T(p) = I(p) = P_{RB}(p) = P_{RK}(p)$, which happens exactly when p is selective,
(4) $P_{RB}(p) = P_{RK}(p)$ and the rest of the classes are mutually different, i.e. p is a semi-P-point and not a Q-point.

An ultrafilter $p \in \omega^*$ such that (1) occurs for p exists in ZFC alone, as essentially proved in [vM]. In Section 4 we will show that the existence of a P-point implies the existence of a semi-P-point which is not a Q-point, hence if there is a p satisfying (3) then there is a q satisfying (4).

It is consistent (it follows from CH or MA) that all four scenarios actually occur. The combination (1),(2),(4) but not (3) is also consistent; it holds in a model obtained from a model of CH by adding $\aleph_1$-many Cohen reals followed by $\aleph_2$-many Random reals, as there are both P-points and Q-points there but no selective ultrafilters (see [Ku]). Another consistent configuration is (1), (4), not (2), and not (3). This happens in any model without Q-points and with P-points, say in the Laver model or in any model of the principle of near coherence of filters (NCF) (see, for example, [Mi]).

The question as to which other configurations are consistent boils down to the following problems:

Question 1.9. Is it consistent with ZFC that:

(1) There are no semi-P-points?
(2) There are Q-points, yet every Q-point is selective?
(3) There are neither Q-points nor semi-P-points?

Recall that it is a well-known open problem whether it is consistent with ZFC that there are no P-points *and* no Q-points.

Next we prove that almost every non-empty difference $M \setminus N$, where M and N are two of the classes $S(p)$, $R(p)$, $T(p)$, $I(p)$, $P_{RB}(p)$, $P_{RK}(p)$, is, in fact, a dense subset of ω^*, the only exception being $R(p) \setminus S(p)$ which is dense in ω^* if and only if p is a rapid ultrafilter.

Theorem 1.10. *(1) For every $p \in \omega^*$,*
(a) $S(p)$ is dense in ω^*.
(b) $I(p) \setminus S(p)$ is dense in ω^*
(c) $T(p) \setminus S(p)$ is dense in ω^*
(2) For every $p \in \omega^*$ which is not a Q-point,
(a) $I(p) \setminus T(p)$ and $T(p) \setminus I(p)$ are dense subsets of ω^*.
(b) $T(p) \setminus R(p)$ is a dense subset of ω^*.
(c) $I(p) \setminus R(p)$ is a dense subset of ω^*.
(d) $P_{RB}(p) \setminus R(p)$, $P_{RB}(p) \setminus I(p)$ and $P_{RB}(p) \setminus T(p)$ are dense in ω^*.
(3) For every $p \in \omega^*$ which is not selective, $P_{RK}(p) \setminus R(p)$, $P_{RK}(p) \setminus I(p)$ and $P_{RK}(p) \setminus T(p)$ are dense subsets of ω^*.
(4) For every $p \in \omega^*$ which is not a semi-P-point, $P_{RK}(p) \setminus P_{RB}(p)$ is a dense subset of ω^*.

Proof. (1.a) Let B be an infinite subset of ω. Then $b \in {}^{\omega\nearrow}\omega$ and $B \in b^\beta(p) \in S(p)$ (recall 0.10).

(1.b) Let B be an infinite subset of ω. Let A be an element of p for which $\omega \setminus A$ is infinite and $0 \notin A$. Let $g : \omega \to \omega$ defined by $g(n) = 0$ if either $n \in A$ and $\{k \in B : k \leq n\} = \emptyset$ or if $n \notin A$, and $g(n) = b_k$ where k is the greatest l such that $b_l < n$. The function g is non-decreasing on $A \in p$, so $q = g^\beta(p) \in I(p)$ (Corollary 0.12). Moreover, $g(A') \subset B$ where $A' = \{n \in A : n \geq b_0\} \in p$, so $B \in q$. On the other hand, $q \notin S(p)$ for if there were $f \in {}^{\omega\nearrow}\omega$ such that $f^\beta(p) = q$, then $D = \{n < \omega : f(n) = g(n)\} \in p$ (Lemma 0.4). Thus, if $m \in D$ then $f(n) = g(n) < n$. This, however, cannot happen for a strictly increasing function f. So, $q \notin S(p)$.

(1.c) Let B' be an infinite subset of ω. Let B be an infinite subset of B' such that $\omega \setminus B$ is infinite. Let A be an element of p. Consider the function $h : B \to A$ defined by $h(b_0) = \min(A \setminus b_0 + 1)$, and $h(b_{n+1}) = \min(A \setminus \max\{h(b_n), b_{n+1}\} + 1)$. Let σ be a permutation of ω extending h^{-1}. Then, $\sigma^\beta(p) \in B^* \cap T(p) \setminus S(p)$.

(2.a) By Theorem 1.7, if p is not a Q-point, then $T(p) \setminus I(p) \neq \emptyset \neq I(p) \setminus T(p)$. By Theorem 1.2, if $r \in T(p) \setminus I(p)$, then $R(r) \subset T(r) \setminus I(p) = T(p) \setminus I(p)$. Hence, $T(p) \setminus I(p)$ is dense in ω^*. The proof that $I(p) \setminus T(p)$ is dense in ω^* is almost identical.

(2.b), (2.c) and (2.d) are consequences of (2.a).

(3) Given the fact that p is not RK-minimal, there is $q \in \omega^*$ such that $q <_{RK} p$. Then $T(q) \subset P_{RK}(p) \setminus T(p)$. As $T(q)$ is dense in ω^*, so is $P_{RK}(p) \setminus T(p)$.

On the other hand, since p is not selective, then $P_{RK}(p) \setminus I(p)$ is non-empty. Take $q \in P_{RK}(p) \setminus I(p)$. By Theorem 1.2.(5), Proposition 1.1 and Theorem 1.2.(8), $R(q) \subset P_{RK}(p) \setminus I(p)$. As $R(q)$ is dense in ω^* the rest follows easily.

(4) If p is not a semi-P-point, there is a $q \in P_{RK}(p) \setminus P_{RB}(p)$. Then $T(q) \subset P_{RK}(p) \setminus P_{RB}(p)$. Thus, $P_{RK}(p) \setminus P_{RB}(p)$ is dense in ω^*. □

Now we are going to analyze when $R(p) \setminus S(p)$ is dense in ω^*. First, we prove that for every $p \in \omega^*$, $p \in \mathrm{Cl}_{\omega^*}(R(p) \setminus S(p))$.

Proposition 1.11. *If $A \in p$, then there is $q \in (R(p) \cap A^*) \setminus S(p)$. That is, for every $p \in \omega^*$, $p \in Cl_{\omega^*}(R(p) \setminus S(p))$.*

Proof. Let $B \subset A$ such that $B \in p$, $0 \notin B$ and $|\omega \setminus B| = \aleph_0$. Define $f : \omega \to \omega$ by letting $f(n) = 0$ if $n \notin B$, and $f(b_{i+1}) = b_i$ for every $i \in \omega$. As f is strictly increasing on $B \in p$, $f^\beta(p) = q \in R(p)$. Moreover, $f[B] \subset A$, thus $q \in A^*$.

On the other hand, if there is $g \in {}^{\omega\nearrow}\omega$ such that $g^\beta(p) = f^\beta(p)$, then $\{n < \omega : g(n) = f(n)\} \in p$. Thus, for some $n < \omega$, $g(n) < n$, which is a contradiction. Hence, $q \notin S(p)$. □

Lemma 1.12. *An element $p \in \omega^*$ is rapid if and only if p satisfies $\mathcal{R}$, where $\mathcal{R}$ is the assertion: For every sequence $d_0 < d_1 < \ldots < d_n < \ldots$ of natural numbers, there is a subsequence $e_0 < e_1 < \ldots < e_n < \ldots$ of $(d_n)_{n<\omega}$, and $A \in p$ such that for every $B \in p$ with $B \subset A$, either $b_0 > e_t$ where $b_0 = a_t$, or there is $n_0 < \omega$ satisfying $b_{n_0+1} - b_{n_0} > e_t - e_s$ where $b_{n_0+1} = a_t$ and $b_{n_0} = a_s$.*

Proof. It is easy to prove that every rapid ultrafilter satisfies the conditions of the theorem. For the converse, assume that an ultrafilter $p \in \omega^*$ satisfies $\mathcal{R}$. We are going to prove that p is rapid. Assume the contrary and let $(d'_n)_{n<\omega}$ be a strictly increasing sequence such that, for every $A \in p$ (see Convention 0.10), $a \not\geq^* d'$. Let d be a function satisfying

$$d_{n+1} - d_n = \Sigma_{m \leq n+1} d'_m + \Sigma_{m<n} d_n. \qquad \#$$

Let $(e_n)_{n<\omega}$ be a subsequence of $(d_n)_{n<\omega}$. For each $m < n$, if $e_m = d_l$ and $e_n = d_k$, then $l < k$, $l \geq m$, $k \geq n$ and $n - m \leq k - l$. So, by (#)

$$e_n - e_m = d_k - d_l \geq d_n \geq d_n - d_m.$$

Let $A \in p$. $A = \{a_n : a_n < d'_n\} \cup \{a_n : a_n \geq d'_n\}$. Note that $C = \{a_n : a_n \geq d'_n\} \notin p$, as for each $n < \omega$, $c_n = a_m \geq d'_m$ and $n \leq m$, so $c_n \geq d'_n$; hence $C \in p$ contradicts our hypothesis on p. Thus, $B = \{a_n : a_n < d'_n\} \in p$ and $b_0 = a_n < d'_n \leq d_n \leq e_n$. Moreover, because of definition of B and (#),

$$b_{n+1} - b_n = a_t - a_s \leq a_t < d'_t \leq d_t - d_{t-1} \leq d_t - d_s \leq e_t - e_s$$

for every $n < \omega$. Therefore, p does not satisfy $\mathcal{R}$. □

Theorem 1.13. *$R(p) \setminus S(p)$ is dense in ω^* if and only if p is rapid.*

Proof. Let D^* be a standard open subset of ω^*. Let $(e_n)_{n<\omega}$ and $A \in p$ witness that p is rapid (see Lemma 1.12). Let $f : \omega \to \omega$ be defined by $f(x) = 0$ if $x \notin A$ and $f(a_i) = e_i$. Hence, $f \in {}^{A\nearrow}\omega$ and $f^\beta(p) \in R(p)$. We are going to show that $f^\beta(p) \notin S(p)$. We will get a contradiction by assuming the contrary: Let $g \in {}^{\omega\nearrow}\omega$ be such that $g^\beta(p) = f^\beta(p)$. Then there is $B \in p$ such that $f \in {}^{B\nearrow}\omega, b_0 \leq f(b_0)$, and for every n,

$$b_{n+1} - b_n \leq f(b_{n+1}) - f(b_n) \qquad *$$

(Corollary 0.12). Moreover, by Lemma 0.5,

$$b_n \leq f(b_n)\ \forall n. \qquad **$$

Let $C = A \cap B$. Since p is rapid, either (1) $c_0 > e_t$ where $c_0 = a_t$, or (2) there is $n_0 < \omega$ such that $c_{n_0+1} - c_{n_0} > e_t - e_s$ where $c_{n_0+1} = a_t, c_{n_0} = a_s$. In the first case, $c_0 = b_l > e_t = f(a_t) = f(b_l)$ for an $l < \omega$, contradicting (**). In the second case, $c_{n_0+1} = b_l$ and $c_{n_0} = b_m$ for some $l > m$. As $e_t = f(a_t) = f(c_{n_0+1}) = f(b_l)$ and $e_s = f(a_s) = f(c_{n_0}) = f(b_m)$,

$$b_l - b_m > f(b_l) - f(b_m). \qquad ***$$

By (*), $b_l - b_m = (b_l - b_{l-1}) + \cdots + (b_{m+1} - b_m) \leq (f(b_l) - f(b_{l-1})) + \cdots + (f(b_{m+1}) - f(b_m)) = f(b_l) - f(b_m)$, which contradicts (***). Therefore, $f^\beta(p) \notin S(p)$. Moreover, $f[A] \subset D$, so $f^\beta(p) \in (R(p) \setminus S(p)) \cap D^*$.

Now, assume that $R(p) \setminus S(p)$ is dense in ω^*, and let $d_0 < d_1 < \cdots < d_n < \ldots$ be a sequence of natural numbers. Fix $q \in (R(p) \setminus S(p)) \cap D^*$. There exist $C \in p$ and $f \in {}^{C\nearrow}\omega$ such that $f^\beta(p) = q$. Since $q \in D^*$, $D \in q$. Moreover, $f[C] \in q$; hence, $D \cap f[C] \in q$, so $f^{-1}(D \cap f[C]) \in p$. Let $A = f^{-1}(D \cap f[C]) \cap C$. Of course, $A \in p$. Put $e_i = f(a_i)$. The sequence $(e_i)_{i<\omega}$ is a subsequence of $(d_i)_{i<\omega}$. Let $B \subseteq A$ be an element of p and assume that (i) $a_l = b_0 \leq e_l$ and for every n,

$$b_{n+1} - b_n \leq e_t - e_s, \qquad \text{ii}$$

where $b_{n+1} = a_t$ and $b_n = a_s$. Define $h \in \omega^\omega$ by $h(k) = 0$ if $k \notin B$ and $h(b_n) = e_t$ if $b_n = a_t$. As $h \restriction B$ coincides with $f \restriction B$ and $f \in {}^{A\nearrow}\omega$, the function h is strictly increasing on B, and $h^\beta(p) = f^\beta(p) = q$. By (i) and (ii), $b_0 \leq h(b_0)$ and $b_{n+1} - b_n \leq h(b_{n+1}) - h(b_n)$ for every n. This means that there is $g \in {}^{\omega\nearrow}\omega$ such that $g^\beta(p) = h^\beta(p) = q$ (Corollary 0.12). This in turn implies that $q \in S(p)$, which is not possible. So, either $a_l = b_0 > e_l$ or there is an $n_0 \in \omega$ such that $|b_{n_0+1} - b_{n_0}| > |e_t - e_s|$ where $b_{n_0+1} = a_t$ and $b_{n_0} = a_s$. Using Lemma 1.12 we conclude that p is rapid. □

2. Semi-P-points and products of ultrafilters

Theorem 1.7 leaves an open question: *Is every semi-P-point a P-point*? We will answer the question in the negative and study the notion of a semi-P-point using products of ultrafilters as introduced by Frolík ([F1]) and Katětov ([Ka]).

Definition 2.1. *For $p, p_n \in \omega^*$ $(n < \omega)$ let*

$$\Sigma_p p_n = \{A \subseteq \omega \times \omega : \{n \in \omega : \{m \in \omega : (n, m) \in A\} \in p_n\} \in p\}.$$

When $p_n = q$ for every $n < \omega$, we write $p \otimes q$ instead of $\Sigma_p p_n$. That is,

$$p \otimes q = \{A \subseteq \omega \times \omega : \{n \in \omega : \{m \in \omega : (n, m) \in A\} \in q\} \in p\}.$$

Remark 2.2. It is easy to see that $\Sigma_p p_n$ is a free ultrafilter on $\omega \times \omega$, hence it can be treated as an ultrafilter in ω^* (via some fixed enumeration of $\omega \times \omega$). What is also immediate is that $\Sigma_p p_n$ is never a P-point, and $p \otimes q$ is never a Q-point. Moreover, if $q_n \in T(p_n)$ (resp., $q_n \leq_{RK} p_n$) for each $n < \omega$, then $\Sigma_p q_n \in T(\Sigma_p p_n)$ (resp., $\Sigma_p q_n \leq_{RK} \Sigma_p p_n$), and $p <_{RK} \Sigma_p p_n$ always holds (see [Bo], [Bl], [vM] and [GFT]). The operation $\otimes$ is associative, non-commutative and without idempotents.

Let us first return to the statement of Lemma 0.4. It required one of the two functions involved to be one-to-one. This is necessary as exhibited by the following example.

Example 2.3. Let $p \in \omega^*$ and let $\pi : \omega \times \omega \longrightarrow \omega$ be the projection on the first coordinate and let π_2 be the projection on the second coordinate. Let $\nabla = \{(n, m) \in \omega \times \omega : m > n\}$. Then (regardless of the choice of p) $\nabla \in p \otimes p$ and $\pi_2 \restriction \nabla$ is a finite-to-one function. Let σ be any finite-to-one extension of $\pi_2 \restriction \nabla$. Trivially, $\pi^\beta(p \otimes p) = \sigma^\beta(p \otimes p) = p$, yet $E_{\pi,\sigma} \notin p \otimes p$. So Lemma 0.4.(2) can fail even if one of the functions is finite-to-one.

Throughout this section π will always denote the projection on the first coordinate and σ a finite-to-one extension of $\pi_2 \restriction \nabla$. For $A \subseteq \omega \times \omega$ let $A_{(n)} = \{m \in \omega : (n, m) \in A\}$ and for $f : \omega \times \omega \longrightarrow \omega$ and $n \in \omega$ let $f_{(n)} : \omega \longrightarrow \omega$ be defined by $f_{(n)}(m) = f((n, m))$. We will also implicitly assume that, unless explicitly stated otherwise, given $p \in \omega^*$ and $f \in \omega^\omega$, $f^\beta(p) \in \omega^*$, in other words, f is not constant on any set in p.

Observe that for $q, p, p_n \in \omega^*$ $(n < \omega)$, $\pi^\beta(\Sigma_p p_n) = p$ and $\sigma^\beta(p \otimes q) = q$.

Proposition 2.4. *Let $p, p_n, q_n \in \omega^*$ $(n < \omega)$.*

(1) If $\{n < \omega : q_n \leq_{RB} p_n\} \in p$, then $\Sigma_p q_n \leq_{RB} \Sigma_p p_n$.
(2) If $\{n < \omega : q_n <_{RB} p_n\} \in p$, then $\Sigma_p q_n <_{RB} \Sigma_p p_n$.
(3) If $p \leq_{RB} p_n$ for all n, then $p <_{RB} \Sigma_p p_n$.

Proof. (1) Let $B = \{n < \omega : q_n \leq_{RB} p_n\}$. For each $n \in B$, there is a finite-to-one function $\sigma_n : \omega \to \omega$ satisfying $\sigma_n^\beta(p_n) = q_n$. Let $\psi : \omega \times \omega \to \omega \times \omega$ be defined by $\psi(n, m) = (n, \sigma_n(m))$ if $n \in B$, and $\psi(n, m) = (n, m)$ if $n \notin B$. Note that the function ψ is finite-to-one.

Claim. $\psi^\beta(\Sigma_p p_n) = \Sigma_p q_n$.

In fact, $A \in \psi^\beta(\Sigma_p p_n)$ if and only if $\{n < \omega : \{m < \omega : (n, m) \in \psi^{-1}(A)\} \in p_n\} \in p$, if and only if $\{n \in B : A_{(n)} \in q_n\} \in p$, if and only if $\{n < \omega : \{m < \omega : (n, m) \in A\} \in q_n\} \in p$, if and only if $A \in \Sigma_p q_n$.

(2) Because of Clause (1), $\Sigma_p q_n \leq_{RB} \Sigma_p p_n$ holds. By hypothesis, $\{n < \omega : q_n <_{RK} p_n\} \supset \{n < \omega : q_n <_{RB} p_n\} \in p$. This, however, implies that $\Sigma_p q_n$ is not equivalent to $\Sigma_p p_n$ (see [Bl]), so $\Sigma_p q_n <_{RB} \Sigma_p p_n$.

(3) By Clause (1), $p \otimes p \leq_{RB} \Sigma_p p_n$. Moreover, $p <_{RB} p \otimes p$ because σ is finite-to-one, $\sigma^\beta(p \otimes p) = p$ and $p <_{RK} p \otimes p$ (Remark 2.2). □

We recall now the definitions of the right and left power of an ultrafilter p, given in [Bo] and [GFT], respectively. For each $1 < \nu < \omega_1$, fix a strictly increasing sequence $(\nu(n))_{n<\omega}$ of ordinals in ω_1 such that

(1) if $1 < \nu < \omega$, $\nu(n) = \nu - 1$;
(2) $\omega(n) = n$ for $n < \omega$;
(3) if ν is a limit ordinal, then $\nu(n) \nearrow \nu$;
(4) if $\nu = \mu + m$ where μ is a limit ordinal and $m < \omega$, then $\nu(n) = \mu(n) + m$ for each $n < \omega$.
(5) if $\nu < \mu$, then $\nu(n) < \mu(n)$ for each $n < \omega$.

Let $p \in \omega^*$. Define the ultrafilters p^α and ${}^\alpha p$ by induction on $\alpha < \omega_1$, as follows: Assume that p^α and ${}^\alpha p$, $\alpha < \nu$ have already been defined. If ν is a limit ordinal, let $p^\nu = \Sigma_p p^{\nu(n)}$ and ${}^\nu p = \Sigma_p({}^{\nu(n)}p)$; if $\nu = \gamma + 1$, set $p^\nu = p^\gamma \otimes p$ and ${}^\nu p = p \otimes {}^\gamma p$.

It is well known that (1) for every $\gamma < \alpha < \omega_1$, $p^\gamma <_{RK} p^\alpha$ and ${}^\gamma p <_{RK} {}^\alpha p$, (2) $p^n = {}^n p$ for every $n \leq \omega$, and (3) $p^{\omega+1} <_{RK} {}^{\omega+1}p$, and also:

Lemma 2.5 ([Bo]). *If* $1 < \nu < \omega_1$, *then* $p^\nu \simeq \Sigma_p p^{\nu(n)}$.

Proposition 2.6. $p^\gamma <_{RB} p^\alpha$ *and* ${}^\gamma p <_{RB} {}^\alpha p$ *for all* $p \in \omega^*$ *and* $1 \leq \gamma < \alpha < \omega_1$.

Proof. Take $1 \leq m < n < \omega$. Let $s = n - m$. In this case (by associativity of $\otimes$) $p^n = p^{s+m} = p^s \otimes p^m$, and $\sigma^\beta(p^s \otimes p^m) = p^m$. As σ is finite-to-one we conclude that $p^m \leq_{RB} p^n$. The strict RB-inequality between p^m and p^n follows from Remark 2.2. Similarly, ${}^m p <_{RB} {}^n p$.

Assume that $p^\lambda <_{RB} p^\delta$ for every $\lambda < \gamma < \omega_1$ and every $\delta < \alpha < \omega_1$ with $0 < \lambda < \delta$, where $\gamma < \alpha$. Because of $\gamma(n) < \alpha(n)$ for every $n < \omega$, and by Proposition 2.4 and Lemma 2.5 we obtain $p^\gamma <_{RB} p^\alpha$.

Now, assume that ${}^\lambda p <_{RB} {}^\delta p$ for every $\lambda < \gamma < \omega_1$ and every $\delta < \alpha < \omega_1$ with $0 < \lambda < \delta$, and $\gamma < \alpha$. We want to demonstrate that ${}^\gamma p <_{RB} {}^\alpha p$. If γ and α are limit ordinals, then the inequality ${}^\gamma p <_{RB} {}^\alpha p$ follows easily from the definition of ${}^\gamma p$, ${}^\alpha p$ and from Proposition 2.4. If $\gamma = \lambda_0 + 1$ and $\alpha = \delta_0 + 1$, then ${}^\gamma p = p \otimes {}^{\lambda_0} p = \Sigma_p({}^{\lambda_0}p) <_{RB} \Sigma_p({}^{\delta_0}p) = p \otimes {}^{\delta_0}p = {}^\alpha p$. If $\gamma = \lambda_0 + 1$ and α is a limit ordinal, then, for some $n_0 < \omega$, $\lambda_0 < \alpha(n)$ for all $n \geq n_0$. Hence, ${}^{\lambda_0+1}p = \Sigma_p({}^{\lambda_0}p) <_{RB} \Sigma_p({}^{\alpha(n)}p) = {}^\alpha p$ (Proposition 2.4). Finally, if γ is limit and $\alpha = \delta_0 + 1$, then $\gamma(n) < \delta_0$. Thus, again by Proposition 2.4, ${}^\gamma p = \Sigma_p({}^{\gamma(n)}p) <_{RB} \Sigma_p({}^{\gamma_0}p) = {}^\alpha p$. □

The last result and Lemma 1.7.(7) in [GFT] produce:

Corollary 2.7. *Let $p \in \omega^*$. For each $0 < \mu < \omega_1$ there are $\gamma, \alpha < \omega_1$ such that $p^\mu \leq_{RB} {}^\alpha p$ and ${}^\mu p \leq_{RB} p^\gamma$.*

Lemma 2.8. *Let $p_n \in \omega^*$ be a semi-P-point for each $n < \omega$, and let $p \in \omega^*$. Let $f : \omega \times \omega \to \omega$ be a function. If there exists $A \in \Sigma_p p_n$ such that for all $n \in \pi[A]$ $f^\beta_{(n)}(p_n) \in \omega^*$, then there is a finite-to-one function $g : \omega \times \omega \to \omega$ such that*

$$g^\beta(\Sigma_p p_n) = f^\beta(\Sigma_p p_n).$$

Proof. Without loss of generality we can assume $A = \omega \times \omega$. As p_n is a semi-P-point for every $n \in \omega$, there is a finite-to-one g_n such that $f^\beta_{(n)}(p_n) = g^\beta_n(p_n)$. Let

$$g((n, m)) = g_n(m).$$

It is easy to see that $g^\beta(\Sigma_p p_n) = f^\beta(\Sigma_p p_n)$, however, g is in general not finite-to-one. We will show, that there is a set $B \in \Sigma_p p_n$ such that $g \restriction B$ is finite-to-one. This obviously suffices.

To that end let for every $n, m \in \omega$, $h_m(n) = \min\{k \in \omega : g_n^{-1}(m) \subseteq k\}$ and let $h : \omega \longrightarrow \omega$ be a function which eventually dominates all h_n, i.e. $\forall n \in \omega$ $|\{m \in \omega : h_n(m) \geq h(m)\}| < \aleph_0$. Let

$$B = \{(n, m) \in \omega \times \omega : m \geq h(n)\}.$$

It is obvious that $B \in \Sigma_p p_n$ and also that $g \restriction B$ is finite to one. □

Theorem 2.9. *Let $p \in \omega^*$ be a semi-P-point. Then, for every $0 < \alpha < \omega_1$, ${}^\alpha p$ and p^α are semi-P-points.*

Proof. Take $\alpha > 1$. Let $f : \omega \times \omega \longrightarrow \omega$ be given. Assuming that ${}^\gamma p$ (resp., p^γ) is a semi-P-point for every $\gamma < \alpha$, we will construct a finite-to-one g so that $f^\beta({}^\alpha p) = g^\beta({}^\alpha p)$ (resp., $f^\beta(p^\alpha) = g^\beta(p^\alpha)$).

There are three possibilities (resp., two possibilities):

Case 1. $\exists A \in {}^\alpha p\ \forall n \in \pi[A]\ f_{(n)} \restriction A_{(n)}$ is constant.

(resp.,

Case 1'. $\exists A \in p^\alpha\ \forall n \in \pi[A]\ f_{(n)} \restriction A_{(n)}$ is constant.)

Case 2. $\alpha = \gamma_0 + 1$ and $\exists A \in {}^\alpha p\ \forall n \in \pi[A]\ f^\beta_{(n)}({}^{\gamma_0} p) \in \omega^*$ (resp., $f^\beta_{(n)}(p^{\gamma_0}) \in \omega^*$).

Case 3. α is a limit ordinal and $\exists A \in {}^\alpha p\ \forall n \in \pi[A]\ f^\beta_{(n)}({}^{\alpha(n)} p) \in \omega^*$.

(resp.,

Case 2'. $\exists A \in {}^\alpha p\ \forall n \in \pi[A]\ f^\beta_{(n)}(p^{\alpha(n)}) \in \omega^*$.)

In Case 1 (resp., Case 1') define $h : \omega \longrightarrow \omega$ by: $h(n) = m$ if $n \in \pi[A]$ and some (any) $k \in A_{(n)}$ $f((n,k)) = m$, and $h(n) = 0$ otherwise. Then $f \restriction A = h \circ \pi \restriction A$. As p is a semi-P-point, there is a finite-to-one $i : \omega \longrightarrow \omega$ such that $h^\beta(p) = i^\beta(p)$. Let $s_\alpha \in Fo(\omega)$ be such that $s_\alpha^\beta({}^\alpha p) = p$ (resp., $s_\alpha^\beta(\Sigma_p p^{\alpha(n)}) = p$). Then let $g = i \circ s_\alpha$. As both i and s_α are finite to one so is g. Also, $\pi({}^\alpha p) = p$ (resp., $\pi(\Sigma_p p^{\alpha(n)}) = p$). Hence, $f^\beta({}^\alpha p) = h^\beta(\pi^\beta({}^\alpha p)) = i^\beta(s_\alpha^\beta({}^\alpha p)) = g^\beta({}^\alpha p)$ (resp., $f^\beta(\Sigma_p p^{\alpha(n)}) = h^\beta(\pi^\beta(\Sigma_p p^{\alpha(n)})) = i^\beta(s_\alpha^\beta(\Sigma_p p^{\alpha(n)})) = g^\beta(\Sigma_p p^{\alpha(n)})$. Therefore, $\Sigma_p p^{\alpha(n)}$ is a semi-P-point. Since $\Sigma_p p^{\alpha(n)} \simeq p^\alpha$, we conclude that p^α is a semi-P-point.)

If Case 2 or Case 3 holds (resp., Case 2'), the existence of the finite-to-one function g for which $f^\beta({}^\alpha p) = g^\beta({}^\alpha p)$ is guaranteed by the inductive hypothesis and by Lemma 2.8, because, in both cases (resp., in this case), ${}^\alpha p$ (resp., p^α) is of the form (resp., is equivalent to an ultrafilter of the form) $\Sigma_p p_n$ where each p_n is a semi-P-point. □

Recall the following standard weakening of the notion of a P-point. Call a free ultrafilter $p \in \omega^*$ a *weak P-point* if it is not an accumulation point of any countable subset of ω^* or, equivalently, for every $X \in [\omega^* \setminus \{p\}]^\omega$ there is an $A \in p \setminus \bigcup X$. It was proved by Kunen [Ku] that weak P-points do exist in ZFC alone. It should be obvious that every P-point is a weak P-point.

Note that for every $p \in \omega^*$, and every $1 < \alpha < \omega_1$, p^α and ${}^\alpha p$ are not weak P-points. To see this consider two cases (the proof for p^α is similar): (1) if $\alpha = \gamma_0 + 1$ let p_n be the ultrafilter (on $\omega \times \omega$) generated by $\{\{n\} \times A : A \in {}^{\gamma_0}p\}$; (2) if α is a limit ordinal, let p_n be the ultrafilter generated by $\{\{n\} \times A : A \in {}^{\alpha(n)}p\}$. In both cases, it is immediate from the definition that $p \in \mathrm{Cl}_{\omega^*}\{p_n : n \in \omega\}$.

Corollary 2.10. *If p is a P-point and $1 < \alpha < \omega_1$, then ${}^\alpha p$ and p^α are semi-P-points which are not weak P-points.*

Proof. ${}^\alpha p$ and p^α are semi-P-points by Theorem 1.7 and Theorem 2.9. That ${}^\alpha p$ and p^α are not weak P-points has been justified before this Corollary. □

Next we will show that non-semi-P-points exist in ZFC.

Lemma 2.11. *Let p be a weak P-point and let $q \not\geq_{RK} p$. Then $p \otimes q$ is not a semi-P-point.*

Proof. We will show that $g^\beta(p \otimes q) \neq p = \pi^\beta(p \otimes q)$ for every finite-to-one g. To that end let $g : \omega \times \omega \longrightarrow \omega$ be finite to one. Let $g_n(m) = g(n,m)$ and let $q_n = g_n^\beta(q)$. As $q \not\geq_{RK} p$, $p \neq q_n$ for every $n \in \omega$, and as p is a weak P-point, there is an $A \in p$ such that $A \notin q_n$ for every $n \in \omega$. Let $B = g^{-1}[A]$. Then

$$B = \bigcup_{n \in \omega} \{n\} \times (g_n^{-1}[A])$$

so $B \notin p \otimes q$. Hence, $g^\beta(p \otimes q) \neq p$. □

An entirely different proof of the fact that there are non-semi-P-points in ZFC can be found in [vM].

As the set of all P-points is downwards closed in the Rudin-Keisler (Rudin-Blass) ordering, it is natural to ask whether the same is true of the class of semi-P-points. As it turns out it is (at least consistently) not true.

Corollary 2.12. *It is consistent that the class of semi-P-points is not downwards closed in the Rudin-Keisler order.*

Proof. Let $p <_{RK} q$ be P-points. By Theorem 2.9, $q \otimes q$ is a semi-P-point but (by Lemma 2.11) $q \otimes p$ is not a semi-P-point. It is easy to see that $q \otimes p \leq_{RK} q \otimes q$. □

Note the curious nature of Lemma 2.11. In order to show that $q \otimes p$ is NOT a semi-P-point we needed p to be a weak P-point. In fact some requirement of this kind is necessary as, for instance, $(p \otimes p) \otimes p$ is a semi-P-point provided that p is a semi-P-point, yet $p \otimes p \not\leq_{RK} p$.

3. Distinguishing indistinguishable ultrafilters

By Theorem 1.2.(1), the set $S(p)$ is always a proper subset of the type $T(p)$. This fact suggests the following natural definition:

Definition 3.1. *For $p, q \in \omega^*$ let $p \trianglelefteq q$ if $\exists f \in \omega^{\nearrow\omega}$ $p = f^\beta(q)$.*

It is easy to see that $\trianglelefteq$ is an ordering (not a pre-ordering) on $T(p)$. Reflexivity was pointed out in Proposition 1.1.(1) and follows from the fact that id is a strictly increasing function; transitivity also holds since a composition of strictly increasing functions is strictly increasing (this was mentioned in Theorem 1.2.(2)) and for antisymmetry it is enough to note that if $f \in \omega^{\nearrow\omega}$ and f^{-1} extends to a strictly increasing function, then we must have $f = id$ (see Theorem 1.2.(3)).

Note that in this new notation $S(p) = \{q \in T(p) : q \trianglelefteq p\}$. As $|S(p)| = \mathfrak{c}$ for every p it follows that there are not $\trianglelefteq$-minimal ultrafilters. The proof that there are no $\trianglelefteq$-maximal ultrafilters is an easy exercise. As in the previous section, Q-points and selective ultrafilters play a prominent role in our investigations.

For $f \in \omega^{\nearrow\omega}$ let f' denote the *derivative* of f defined by $f'(n) = f(n+1) - f(n)$. For $f, g \in \omega^\omega$ let $\Delta_{f,g}(n) = |f(n) - g(n)|$. Now, for $f, g \in \omega^{\nearrow\omega}$ let

$$f \preccurlyeq g \quad \text{if} \quad \exists h \in \omega^{\nearrow\omega} \quad g = h \circ f.$$

It is easy to verify that $\preccurlyeq$ is a partial order on $\omega^{\nearrow\omega}$ with the least element id. Let $\leq$ denote the standard (pointwise) ordering on ω^ω.

Lemma 3.2. *Let $f, g \in \omega^{\nearrow\omega}$. Then the following are equivalent:*

(1) $f \preccurlyeq g$,
(2) $f \leq g$ and $f' \leq g'$,
(3) $f \leq g$ and $\Delta_{f,g}$ is non-decreasing.

Proof. (1)⇒(2): $f \leq g$ as $g(n) = h(f(n)) \geq f(n)$ for some $h \in \omega^{\nearrow\omega}$ and every $n \in \omega$. Now (for the same h), $g'(n) = h(f(n+1)) - h(f(n)) = h(f(n) + (f(n+1) - f(n))) - h(f(n)) \geq h(f(n)) + f(n+1) - f(n) - h(f(n)) = f'(n)$.

(2)⇒(3): This is completely trivial.

(3)⇒(1): Assuming (3) let

$$h(i) = \begin{cases} i & \text{if } i < f(0) \\ g(k) + i - f(k) & \text{if } i \in [f(k), f(k+1)) \end{cases}$$

Then $h \in \omega^{\nearrow\omega}$ and $g(k) = h(f(k))$ for every $k \in \omega$ as required. □

Definition 3.3. *For $p \in \omega^*$ and $f, g \in \omega^{\nearrow\omega}$ define:*

(1) $f \preccurlyeq_p g$ if $\exists h \in \omega^{\nearrow\omega}$ $\{n \in \omega : g(n) = h(f(n))\} \in p$,
(2) $f \approx_p g$ if $\{n \in \omega : g(n) = f(n)\} \in p$.

The reason for introducing the pre-ordering $\preccurlyeq_p$ (the routine verification that it is indeed a pre-ordering is omitted) is the following:

Proposition 3.4. *Let $p \in \omega^*$. Then $(S(p), \trianglelefteq)$ is anti-isomorphic to the (quotient) order $(\omega^{\nearrow\omega}, \preccurlyeq_p)$.*

Proof. Define $\Phi : \omega^{\nearrow\omega} \longrightarrow S(p)$ by $\Phi(f) = f^\beta(p)$. Then:

(1) $\Phi(f) = \Phi(g)$ if and only if $f \approx_p g$,
(2) $S(p) = \text{rng}(\Phi)$,
(3) $\Phi(f) \trianglelefteq \Phi(g)$ if and only if $g \preccurlyeq_p f$.

We will only check (3) as the rest is even easier. $\Phi(f) \trianglelefteq \Phi(g)$ if and only if $\exists h \in \omega^{\nearrow\omega}$ $h^\beta(\Phi(g)) = \Phi(f)$ if and only if $h^\beta(g^\beta(p)) = f^\beta(p)$ if and only if $\{n \in \omega : f(n) = h(g(n))\} \in p$ if and only if $g \preccurlyeq_p f$. □

So studying the order $\trianglelefteq$ is (at least locally) equivalent to studying $\preccurlyeq_p$ for the appropriate $p \in \omega^*$. Note the subtle difference between the ordering $\preccurlyeq_p$ and the standard $\leq_p$. While $\leq_p$ is a linear order for every $p \in \omega^*$ it is not necessarily true for $\preccurlyeq_p$. However, the following is true for $\preccurlyeq_p$:

Proposition 3.5. $(\omega^{\nearrow\omega}, \preccurlyeq_p)$ *is upwards directed for every $p \in \omega^*$.*

Proof. Note that to prove this it is enough to show that $\preccurlyeq$ is upwards directed. Given $f, g \in \omega^{\nearrow\omega}$ find an $h \in \omega^{\nearrow\omega}$ such that $f \leq h$ and $f' \leq h'$, $g \leq h$ and $g' \leq h'$ (let for instance $h = f + g$). Then by Lemma 3.2, $f \preccurlyeq h$ and $g \preccurlyeq h$, hence $f \preccurlyeq_p h$ and $g \preccurlyeq_p h$ for every $p \in \omega^*$ □

Corollary 3.6. $(S(p), \trianglelefteq)$ *is downwards directed for every $p \in \omega^*$.*

Proof. This statement follows directly from the previous two propositions. □

Next we want to show that in some cases $\preccurlyeq_p$ is a linear order. To that end we need an analog of Lemma 3.2 for $\preccurlyeq_p$.

Lemma 3.7. *Let $p \in \omega^*$ and $f, g \in \omega^{\nearrow\omega}$. Then $f \preccurlyeq_p g$ if and only if $\exists A \in p$ $f \restriction A \leq g \restriction A$ and $\Delta_{f,g} \restriction A$ is non-decreasing.*

Proof. For the direct implication assume that $f \preccurlyeq_p g$ and let $h \in \omega^{\nearrow\omega}$ be such that $A = \{n \in \omega : g(n) = h(f(n))\} \in p$. Then:

$g(n) = h(f(n)) \geq f(n)$ for every $n \in A$, hence $f \restriction A \leq g \restriction A$.

Now, for $m < n \in A$, $\Delta_{f,g}(n) = g(n) - f(n) = h(f(n)) - f(n) = h(f(m) + (f(n) - f(m))) - f(n) \geq h(f(m)) + (f(n) - f(m)) - f(n) = h(f(m)) - f(m) = \Delta_{f,g}(m)$.

For the reverse implication let f, g, A be given. Enumerate $A = \{a_i : i \in \omega\}$ in an increasing manner and let:

$$h(i) = \begin{cases} i & \text{if } i < f(a_0) \\ g(a_k) + i - f(a_k) & \text{if } i \in [f(a_k), f(a_{k+1})) \end{cases}$$

Obviously, $g(a_k) = h(f(a_k))$ and $h \in \omega^{\nearrow\omega}$ follows easily as $g(a_{k+1}) \geq g(a_k) + f(a_{k+1}) - f(a_k)$. □

Proposition 3.8. *$p \in \omega^*$ is selective if and only if $\preccurlyeq_p$ is linear.*

Proof. For the direct implication assume that p is selective. By Lemma 3.7, it is enough to show that $\forall f, g \in \omega^{\nearrow\omega}$ $\exists A \in p$ such that $f \restriction A \leq g \restriction A$ and $\Delta_{f,g} \restriction A$ is non-decreasing, or vice versa.

As $\omega = \{n \in \omega : f(n) \leq g(n)\} \cup \{n \in \omega : g(n) \leq f(n)\}$ we can assume that $A_0 = \{n \in \omega : f(n) \leq g(n)\} \in p$ (the other case is completely analogous). Now, consider $\Delta_{f,g}$ and let for $m < n \in \omega$

$$\phi(\{m, n\}) = \begin{cases} 0 & \text{if } \Delta_{f,g}(m) \leq \Delta_{f,g}(n) \\ 1 & \text{if } \Delta_{f,g}(m) > \Delta_{f,g}(n) \end{cases}$$

As every selective ultrafilter is Ramsey, there is an $A_1 \in p$ such that $|\phi''([A_1]^2)| = 1$, i.e. A_1 is homogeneous. As homogeneity in color 1 would produce a strictly decreasing sequence of non-negative integers (which is absurd), A_1 is homogeneous in color 0, hence $\Delta_{f,g} \restriction A_1$ is non-decreasing. Then $A = A_0 \cap A_1$ is as required.

We will prove the reverse implication in two steps. First we will show that if $\preccurlyeq_p$ is linear then p is a Q-point. To that end let $\{I_n : n \in \omega\}$ be a partition of ω into finite sets. Let $f, g \in \omega^{\nearrow\omega}$ be such that: $f \leq g$ and $\Delta_{f,g}$ is strictly decreasing on I_n for every $n \in \omega$. To construct such f and g is easy. Now, as $\preccurlyeq_p$ is linear $f \preccurlyeq_p g$, and by the previous lemma, $\Delta_{f,g}$ is non-decreasing on a set $A \in p$. Then, of course, $|A \cap I_n| \leq 1$ for every $n \in \omega$, hence p is a Q-point.

To show that p is in fact selective it is sufficient to show that p is $\leq_{RK}$-minimal; in other words, for every $f \in \omega^\omega$ there is an $A \in p$ such that $f \restriction A$ is constant or one-to-one. Let an $f \in \omega^\omega$ be given. Construct $g, h \in \omega^{\nearrow\omega}$ such that $g \leq h$ and $f = \Delta_{g,h}$. Again this task is easy to fulfill. By linearity of $\preccurlyeq_p$ there is a set $B \in p$ such that $f \restriction B = \Delta_{g,h} \restriction B$ is non-decreasing. Then, either $f \restriction B$ is eventually constant in which case let $A = B \setminus n$, where $\forall i, j \geq n$ $f(i) = f(j)$, or $f \restriction B$ is finite-to-one in which case an application of the fact that p is a Q-point produces $A \in p$ such that $f \restriction A$ is strictly increasing, hence one-to-one. □

Now we are ready to return to the study of $\trianglelefteq$.

Proposition 3.9. *For every $p \in \omega$ $(R(p), \trianglelefteq)$ is upwards directed and downwards directed.*

Proof. Theorem 1.2.(6) and Theorem 1.2.(8) imply that $(R(p), \trianglelefteq)$ is downwards directed. Now, let $q = f^\beta(p)$ and $q' = g^\beta(p)$, and let $A \in p$ be such that both $f \restriction A$ and $g \restriction A$ are strictly increasing. Let a be the increasing enumeration of A. Let h be an extension of a^{-1} to ω, and let $r = h^\beta(p)$. Then both $f \circ a$ and $g \circ a$ are strictly increasing and $q = f^\beta(a^\beta(r))$ and $q' = g^\beta(a^\beta(r))$; hence r is a common upper bound for q and q'. □

Corollary 3.10. *Let $p \in \omega^*$. Then the following are equivalent:*

(1) p is a Q-point,
(2) $(T(p), \trianglelefteq)$ is upwards directed,
(3) $(T(p), \trianglelefteq)$ is downwards directed.

Proof. If p is a Q-point, then $T(p) = R(p)$, so, by Proposition 3.9, (1) implies (2) and (3).

If $(T(p), \trianglelefteq)$ is upwards directed then it is downwards directed by Corollary 3.6.

To finish the proof assume that $(T(p), \trianglelefteq)$ is downwards directed and let $\{I_n : n \in \omega\}$ be a partition of ω into finite sets. Let σ be a permutation on ω strictly decreasing on each I_n. Let $q = \sigma^\beta(p)$. As $(T(p), \trianglelefteq)$ is downwards directed there are $h, g \in \omega^{\nearrow\omega}$ such that $h^\beta(p) = g^\beta(q) = g^\beta(\sigma^\beta(p))$. By Lemma 0.4.(2), $E_{h,g\circ\sigma} \in p$, and $\sigma \restriction E_{h,g\circ\sigma}$ is strictly increasing, hence $|I_n \cap E_{h,g\circ\sigma}| \leq 1$ for every $n \in \omega$, and therefore p is a Q-point. □

Corollary 3.11. $(T(p), \trianglelefteq)$ *is linear if and only if p is selective.*

Proof. Follows immediately from Proposition 3.8 and Corollary 3.10. □

Note that if p is not a Q-point then $(T(p), \trianglelefteq)$ decomposes into downwards directed components $R(q)$, $p \in T(p)$. The natural questions one would ask are:

(1) What are the possibilities for the number of components of $(T(p), \trianglelefteq)$?
(2) What are the possible cofinalities (coinitialities) of $(T(p), \trianglelefteq)$?
(3) What are the possible lengths of decreasing (increasing) chains in $(T(p), \trianglelefteq)$?

It is not difficult to see that $(T(p), \trianglelefteq)$ always contains a chain of length $\mathfrak{b}$, where $\mathfrak{b}$ is the minimal length of an unbounded chain in ω^ω ordered by eventual dominance. Similarly, the coinitiality of $(T(p), \trianglelefteq)$ lies between $\mathfrak{b}$ and $cof(\mathfrak{d})$ for any selective ultrafilter p. Here $\mathfrak{d}$ denotes the dominating number of ω^ω.

4. $S(p)$-pseudocompact spaces and the class of Frolík

Recall that, given a topological space X, a sequence $(U_n)_{n<\omega}$ of open subsets of X is a *Frolík sequence* if for each filter $\mathcal{G}$ of infinite subsets of ω,

$$\bigcap_{F\in\mathcal{G}} \mathrm{cl}_X(\bigcup_{n\in F} U_n) \neq \emptyset.$$

The *Frolík class* $\mathcal{F}$ is the class of productively pseudocompact spaces (i.e, the class of pseudocompact spaces whose product with every pseudocompact space is also pseudocompact). Theorem 3.6 of [F1] shows that a pseudocompact space X belongs to $\mathcal{F}$ if and only if each infinite family of pairwise disjoint open subsets of X contains a subfamily $(U_n)_{n<\omega}$ which is a Frolík sequence. It is known that a Frolík space is not necessarily p-pseudocompact for some $p \in \omega^*$. On the other hand, in [ST] it was proved that if X is Frolík, then it is $P_{RK}(p)$-pseudocompact for every $p \in \omega^*$. We are going to strengthen this result by proving that every space in $\mathcal{F}$ is $S(p)$-pseudocompact for every $p \in \omega^*$. First we need a lemma, the proof of which is left to the reader.

Lemma 4.1. *Let $g : \omega \to \omega$ be a function. Then:*

(1) The following assertions are equivalent.
  (a) There exists $f \in \omega^\omega$ such that $g \circ f \in {}^{\omega\nearrow}\omega$.
  (b) There exists an infinite subset N of ω such that $g \in {}^{N\nearrow}\omega$.
  (c) $|g[\omega]| = \aleph_0$.

(2) There is $h : \omega \to \omega$ such that $h \circ g$ is strictly increasing if and only if g is one-to-one.

(3) If $(U_n)_{n<\omega}$ is a Frolík sequence, and $g : \omega \to \omega$ is a one-to-one function, then $(U_{g(n)})_{n<\omega}$ is a Frolík sequence.

Theorem 4.2. *Each space X in the class of Frolík $\mathcal{F}$ is $S(p)$-pseudocompact for every $p \in \omega^*$.*

Proof. Let $p \in \omega^*$ and let $(U_n)_{n<\omega}$ be a sequence of pairwise disjoint open sets of X. Since $X \in \mathcal{F}$, there exists $g : \omega \to \omega$, an injective function, such that $(U_{g(n)})_{n<\omega}$ is a Frolík sequence. In view of the definition of a Frolík sequence, any rearrangement of $(U_{g(n)})_{n<\omega}$ is again a Frolík sequence; so, g can be taken as a strictly increasing function (Lemma 4.1). Take $x \in X$ such that

$$x \in \bigcap_{F\in p} cl_X \left(\bigcup_{k\in F} U_{g(n)}\right) \neq \emptyset.$$

This means that for each neighborhood V of x and for each $F \in p$, there is $k = k_F \in F$ such that $V_k \cap V \neq \emptyset$ where $V_k = U_{g(k)}$. If $H = \{k_F : F \in p\} \notin p$, then $\omega \setminus H \in p$, thus $k_{\omega\setminus H}$ is an element of both H and $\omega \setminus H$, which is not possible; so $H \in p$, and this implies that $x = p\text{-}\lim(U_{g(n)}) = g^\beta(p)\text{-}\lim(U_n)$ (Lemma 0.9). Since g is strictly increasing, we have proved what we wanted. □

Letting $X = \prod_{p\in\omega^*}(\beta(\omega) \setminus \{p\})$ produces a Frolík space, hence $S(p)$-pseudocompact for every $p \in \omega^*$, which is not q-pseudocompact for any $q \in \omega^*$ (see Example 2.9 in [ST1]).

The subclass $\mathcal{F}^*$ of $\mathcal{F}$ is defined as the class of spaces X with the property that each sequence of disjoint open sets in X has a subsequence such that each of its elements meets some fixed compact set. This class was introduced and studied by N. Noble in [N]. In particular, Noble showed that $X \in \mathcal{F}^*$ whenever the set X endowed with the weak topology induced by the real-valued functions on X which are continuous on all compact subsets of X, $k_R X$, is pseudocompact. Thus, pseudocompact spaces which are locally compact or sequential are $S(p)$-pseudocompact for every $p \in \omega^*$. A space X is a *k_R-space* if $X = k_R X$. Noble also proved in [N] that every completely regular space can be embedded as a closed subspace of a pseudocompact k_R-space. Hence:

Theorem 4.3. *Every pseudocompact space can be embedded as a closed subspace of a space which is $S(p)$-pseudocompact for every $p \in \omega^*$. So, if $M \supset S(p)$, M-pseudocompactness is not inherited by closed subsets.*

Problem 4.4. Give an example of a space which is $S(p)$-pseudocompact for every $p \in \omega^*$ and does not belong to $\mathcal{F}$.

In the same vein we can characterize the $S(p)$-pseudocompact spaces having all of their closed subsets sharing this property. We omit the proof because it is similar to that given for Theorem 2.8 in [ST1].

Theorem 4.5. *Let X be a topological space and let M be one of the sets $S(p)$, $R(p)$, $T(p)$, $P_{RB}(p)$, $P_{RK}(p)$. Then, every closed subset of X is M-pseudocompact if and only if X is M-compact.*

Using a similar demonstration to that given for Theorem 2.10 in [ST1] we obtain:

Theorem 4.6. *Let $p \in \omega^*$ and M be one of the sets $S(p)$, $R(p)$, $T(p)$, $P_{RB}(p)$ or $P_{RK}(p)$. Then, a pseudocompact space X is M-pseudocompact if and only if it is locally M-pseudocompact.*

Corollary 4.7. *Let $p \in \omega^*$ and M one of the sets $S(p)$, $R(p)$, $T(p)$, $P_{RB}(p)$ or $P_{RK}(p)$. Then, each open pseudocompact subset of an M-pseudocompact space is M-pseudocompact.*

Corollary 4.8. *Let $p \in \omega^*$ and M one of the sets $S(p)$, $R(p)$, $T(p)$, $P_{RB}(p)$ or $P_{RK}(p)$. Then, a free topological sum $X = \bigoplus_{\alpha\in A} X_\alpha$, where $X_\alpha \neq \emptyset$, is M-pseudocompact if and only if each X_α is M-pseudocompact and $|A| < \aleph_0$.*

5. $R(p)$-pseudocompactness of subspaces of $\beta(\omega)$

Given a collection $\mathcal{M}$ of elements of ω^ω and $p \in \omega^*$, we will denote by $M(p)$ the set $\{f^\beta(p) : f \in \mathcal{M}\}$. A subcollection $\mathcal{M}$ of ω^ω is a *p-si-ideal*, for a $p \in \omega^*$, if $id \in \mathcal{M}$ and for each $g \in \mathcal{M}$ and each $f \in {}^{\omega\nearrow}\omega$, $(f \circ g)^\beta(p) \in M(p)$. We will say that $\mathcal{M}$ is a *strong-p-si-ideal* if $id \in \mathcal{M}$ and for every (ψ, A, f)

$\in \mathcal{M} \times p \times P({}^{\omega\nearrow}\omega)$ with $\psi[A] \subset \mathrm{dom}(f)$, there exist $g \in \mathcal{M}$ and $B \in p$ such that $g \restriction B = f \circ (\psi \restriction B)$.

Each strong-p-si-ideal is a p-si-ideal. The collections $\bigcup\{{}^{A\nearrow}\omega : A \in p\}$, $Sym(\omega)$, $Fo(\omega)$, $Nd(\omega)$ and ω^ω are strong-p-si-ideals for each $p \in \omega^*$, and ${}^{\omega\nearrow}\omega$ is a p-si-ideal which is not a strong-p-si-ideal for every $p \in \omega^*$ (see Lemma 0.11 and Corollary 0.12).

In [GF] it was proved that for $p \in \omega^*$ and $\omega \subset X \subset \beta(\omega)$, X is p-pseudocompact if and only if $P_{RK}(p) \subset X$. Moreover, in [ST] the proposition: a subset X of $\beta(\omega)$ containing ω is $P_{RK}(p)$-pseudocompact if and only if $X \cap P_{RK}(p)$ is dense in ω^*, was proved. We provide an analogous result for $R(p)$, $I(p)$, $T(p)$ and $P_{RB}(p)$. The proof of this assertion will be given by demonstrating several lemmas.

Lemma 5.1. *Let M be a subset of ω^*. A subset X of $\beta\omega$ containing ω is M-pseudocompact if and only if for every one-to-one function $f \in \omega^\omega$, there is $p \in M$ such that $f^\beta(p) \in X$.*

Proof. Assume that X is M-pseudocompact and let $f \in \omega^\omega$ be a one-to-one function. $(\{f(n)\})_{n<\omega}$ is a sequence of disjoint open subsets of X. So, there are $x \in X$ and $p \in M$ such that $x = p\text{-lim}(f(n))$. This means $f^\beta(p) = x \in X$ (Lemma 0.6).

The converse implication is also true because if $(A_n)_{n<\omega}$ is a sequence of disjoint open subsets of X, we can choose a point $a_n \in A_n \cap \omega$ for each $n < \omega$. The function f defined by $f(n) = a_n$ is a one-to-one function; so, there is $p \in M$ such that $x = f^\beta(p) \in X$. Thus, x is a p-limit point of $(A_n)_{n<\omega}$. □

Lemma 5.2. *Let $X \subset \beta(\omega)$ with $\omega \subset X$, $p \in \omega^*$ and let $\mathcal{M} \subset \omega^\omega$ be a p-si-ideal. If X is $M(p)$-pseudocompact, then $X \cap (M(p) \cup \omega)$ is $M(p)$-pseudocompact.*

Proof. Let $g : \omega \to \omega$ be a one-to-one function. There is an infinite set $T \subseteq \omega$ such that $g \restriction T$ is a strictly increasing function. Consider the sequence $((g \circ t)(n))_{n<\omega}$ (see Convention 0.10) and note that $g \circ t$ is a strictly increasing function from ω to ω. As X is $M(p)$-pseudocompact, there exist $x \in X$ and $r \in M(p)$ such that $(g \circ t)^\beta(r) = x$. As $r \in M(p)$, there is a function $h \in \mathcal{M}$ such that $h^\beta(p) = r$. So:

$$(g \circ t \circ h)^\beta(p) = (g \circ t)^\beta(h^\beta(p)) = (g \circ t)^\beta(r) = x.$$

Let $s = (t \circ h)^\beta(p)$. As t and $g \circ t$ are strictly increasing, $h \in \mathcal{M}$ and $\mathcal{M}$ is a p-si-ideal, then s and x are elements of $M(p)$. Moreover, $g^\beta(s) = x$. That is, $x = s\text{-lim}(g(n))$. □

Lemma 5.3. *Let $X \subset \beta(\omega)$ with $\omega \subset X$, $p \in \omega^*$ and M a subset of $\beta(\omega)$ be given. If $X \cap (M \cup \omega)$ is M-pseudocompact, then $(X \cap M) \setminus \omega$ is dense in ω^*.*

Proof. Assume that $X \cap (M \cup \omega)$ is M-pseudocompact. Let A be an infinite subset of ω. We are going to check that $A^* \cap X \cap M \neq \emptyset$. By the hypothesis there are $q \in M$ and $x \in X \cap (M \cup \omega)$ such that $x = q\text{-lim}(a(n))$. So, $a^\beta(q) = x$. (Note that x must belong to ω^* as a is one-to-one.)

To see that x is also an element of A^* let $B \in x$. Since $a^\beta(q) = x$, then $a^{-1}(B) \in q$. Thus $a^{-1}(B) \neq \emptyset$, then $B \cap A \neq \emptyset$. But this is true for all $B \in x$, so $A \in x$. So, $x \in A^*$. □

Lemma 5.4. *Let $X \subset \beta(\omega)$ with $\omega \subset X$, $p \in \omega^*$ and a strong-p-si-ideal $\mathcal{M}$ be given. If $(X \cap M(p)) \setminus \omega$ is dense in ω^*, then X is $M(p)$-pseudocompact.*

Proof. Let $g \in \omega^\omega$ be a one-to-one function. Say $g(n) = x_n$. We will find $r_g \in M(p)$ such that $g^\beta(r_g) \in X$. There is an infinite subset T of ω on which g is strictly increasing. Denote by f the composition $g \circ t$; that is, $f(n) = x_{t(n)}$. Of course, f and t are elements of ${}^{\omega\nearrow}\omega$. Consider the set $A = \{f(n) : n < \omega\}$. By the hypothesis there is $x_g \in X \cap M(p) \cap A^*$. Let ψ be an element of $\mathcal{M}$ such that $\psi^\beta(p) = x_g$. Now, take the set $\psi^{-1}(A)$, which is infinite as $A \in x_g$ and $\psi^\beta(p) = x_g$, so $\psi^{-1}(A) \in p \in \omega^*$. Let $\phi : \omega \to \omega$ be defined by $\phi(n) = m$ if $\psi(n) = f(m)$, and $\phi(n) = 0$ if $n \notin \psi^{-1}(A)$. The function $\phi \restriction \psi^{-1}(A)$ is equal to $f^{-1} \circ (\psi \restriction \psi^{-1}(A))$. As $\psi \in \mathcal{M}$, $f^{-1} \in P({}^{\omega\nearrow}\omega)$, $\psi^{-1}(A) \in p$, and $\mathcal{M}$ is a strong-p-si-ideal, there exists $\chi \in \mathcal{M}$ such that $r_g = \phi^\beta(p) = \chi^\beta(p) \in M(p)$ (see Lemma 0.4.(1)).

Claim 1. $x_g = r_g\text{-}\lim(f(n)) = r_g\text{-}\lim(x_{t(n)})$.

Let $B \in x_g$. We have to show that $f^{-1}(B) \in r_g$. Since $\phi^\beta(p) = r_g$, it is enough to prove that $\phi^{-1}f^{-1}(B) \in p$. In order to do this we prove:

Claim 2. $\phi^{-1}f^{-1}(B \cap A) \supset \psi^{-1}(B \cap A)$.

Let $n \in \psi^{-1}(B \cap A)$. So, $\psi(n) = f(m)$ for some $m \in \omega$. This means that $\phi(n) = m$. Then $f(\phi(n)) = f(m) \in B \cap A$. So, $\phi(n) \in f^{-1}(B \cap A)$. Therefore, $n \in \phi^{-1}f^{-1}(B \cap A)$ and Claim 2 follows.

Since $B \cap A \in x_g = \psi^\beta(p)$, $\psi^{-1}(B \cap A) \in p$. So, $\phi^{-1}f^{-1}(B) \in p$. Therefore $x_g = r_g\text{-}\lim(f(n))$ and the proof of Claim 1 is finished.

By Lemma 0.9 this last equality means that $x_g = q\text{-}\lim(x_n)$ where $q = t^\beta(r_g)$. So, $q = t^\beta(\chi^\beta(p))$. However, $t \in {}^{\omega\nearrow}\omega$, $\chi \in \mathcal{M}$ and $\mathcal{M}$ is a strong-p-si-ideal, so $q \in M(p)$. □

This sequence of lemmas produces the following four theorems.

Theorem 5.5. *Let $p \in \omega^*$ and $X \subset \beta(\omega)$ with $\omega \subset X$. Let $\mathcal{M}$ be a strong-p-si-ideal. Then, the following are equivalent.*

(1) X is $M(p)$-pseudocompact.
(2) $X \cap (M(p) \cup \omega)$ is $M(p)$-pseudocompact.
(3) $X \cap M(p) \setminus \omega$ is dense in ω^*.

As $\bigcup\{{}^{A\nearrow}\omega : A \in p\}$, $Sym(\omega)$, $Fo(\omega)$, $Nd(\omega)$ and ω^ω are strong-p-si-ideals ($p \in \omega^*$), we obtain:

Theorem 5.6. *Let $p \in \omega^*$. Let M be one of the sets $R(p)$, $I(p)$, $T(p)$, $P_{RB}(p)$ or $P_{RK}(p)$ and let $X \subset \beta(\omega)$ with $\omega \subset X$. Then, the following are equivalent.*

(1) X is M-pseudocompact.
(2) $X \cap (M \cup \omega)$ is M-pseudocompact.
(3) $X \cap M \setminus \omega$ is dense in ω^*.

In particular, $R(p) \cup \omega$ (resp., $I(p)$, $T(p) \cup \omega$, $P_{RB}(p)$, $P_{RK}(p)$) is an example of an $R(p)$-pseudocompact (resp., $I(p)$-pseudocompact, $T(p)$-pseudocompact, $P_{RB}(p)$-pseudocompact, $P_{RK}(p)$-pseudocompact) space.

Theorem 5.7. *Let $p \in \omega^*$, $X \subset \beta(\omega)$ with $\omega \subset X$, and let $\mathcal{M} \subset \omega^\omega$ be a p-si-ideal. Then, if X is $M(p)$-pseudocompact, then $X \cap (M(p) \cup \omega)$ is $M(p)$-pseudocompact. In particular, $X \cap M(p)$ is dense in ω^*.*

Theorem 5.8. *Let $p \in \omega^*$. Let $\omega \subset X \subset \beta(\omega)$. If X is $S(p)$-pseudocompact, then $X \cap (S(p) \cup \omega)$ is $S(p)$-pseudocompact and $X \cap S(p)$ is dense in ω^*.*

Proposition 5.9. *Let X be a subset of $\beta(\omega)$ containing ω. If $X \supset S(p)$, then X is an $S(p)$-pseudocompact space.*

Proof. Let $f \in \omega^\omega$ be one-to-one. There is an infinite $T \subseteq \omega$ such that $f \restriction T$ is strictly increasing. So, $r = t^\beta(p) \in S(p)$. Moreover, $f \circ t$ is strictly increasing, so $q = (f^\beta \circ t^\beta)(p) \in S(p)$. By Lemma 0.6 $q = r\text{-}\lim(f(n))$. □

Problems 5.10. (1) Are the propositions in Theorem 5.7 equivalent?
(2) Is the converse in Proposition 5.9 true?

In the following examples, Q_1, Q_2, Q_3, Q_4 denote the set of non-rapid ultrafilters, Q-points, selective ultrafilters and semi-P-points in ω^*, respectively.

Examples 5.11. (1) There is a space X_1 which is $R(p)$-pseudocompact for every $p \in \omega^* \setminus Q_1$ and it is not $S(p)$-pseudocompact for any $p \in \omega^* \setminus Q_1$.
(2) There is a space X_2 which is $T(p)$-pseudocompact for every $p \in \omega^* \setminus Q_2$ and it is not $I(p)$-pseudocompact for any $p \in \omega^* \setminus Q_2$.
(3) There is a space X_3 which is $I(p)$-pseudocompact for every $p \in \omega^* \setminus Q_2$ and it is not $T(p)$-pseudocompact for any $p \in \omega^* \setminus Q_2$.
(4) There is a space X_4 which is $P_{RK}(p)$-pseudocompact for every $p \in \omega^* \setminus Q_3$ and it is not $T(p)$-pseudocompact for any $p \in \omega^* \setminus Q_3$.
(5) There is a space X_5 which is $P_{RK}(p)$-pseudocompact for every $p \in \omega^* \setminus Q_3$ and it is not $I(p)$-pseudocompact for any $p \in \omega^* \setminus Q_3$.
(6) There is a space X_6 which is $P_{RK}(p)$-pseudocompact for every $p \in \omega^* \setminus Q_4$ and it is not $P_{RB}(p)$-pseudocompact for any $p \in \omega^* \setminus Q_4$.
(7) There is a space X_7 which is $S(p)$-pseudocompact for every $p \in \omega^*$ and it is not p-pseudocompact for any $p \in \omega^*$.

Proof. For $i \in \{1, 2, 3, 4\}$, let K_i be the one-point compactification of the space $\bigoplus_{p\in\omega^*\setminus Q_i} \beta\omega_p$ where $\beta\omega_p$ is a copy of $\beta\omega$ for every $p \in \omega^* \setminus Q_i$, and denote by ∞_i the point which compactifies $\bigoplus_{p\in\omega^*\setminus Q_i} \beta\omega_p$ in K_i. Since M-pseudocompactness is inherited by regular closed sets, and using Theorem 5.6, we easily get that subspace $X_1 = \{\infty_1\} \cup \bigoplus_{p\in\omega^*\setminus Q_1} (\beta\omega_p \setminus I(p))$, subspaces $X_2 = \{\infty_2\} \cup \bigoplus_{p\in\omega^*\setminus Q_2} (\beta\omega_p \setminus I(p))$ and $X_3 = \{\infty_2\} \cup \bigoplus_{p\in\omega^*\setminus Q_2} (\beta\omega_p \setminus T(p))$ of K_2, subspaces $X_4 = \{\infty_3\} \cup \bigoplus_{p\in\omega^*\setminus Q_3} (\beta\omega_p \setminus T(p))$ and $X_5 = \{\infty_3\} \cup \bigoplus_{p\in\omega^*\setminus Q_3} (\beta\omega_p \setminus I(p))$ of K_3, and subspace $X_6 = \{\infty_4\} \cup \bigoplus_{p\in\omega^*\setminus Q_4} (\beta\omega_p \setminus P_{RB}(p))$ of K_4, satisfy the requirements.

Space $X_7 = \prod_{p\in\omega^*} (\beta\omega \setminus \{p\})$ is $S(p)$-pseudocompact space for every $p \in \omega^*$ but it is not q-pseudocompact for any $q \in \omega^*$ (see Example 2.9 in [ST]). □

On the other hand, there are some wide classes of spaces where all properties considered in our discussion in this article coincide: A space X is *ultracompact* (resp., *ultrapseudocompact*) if every sequence of points (resp., every sequence of non empty open sets) in X has a q-limit for every $q \in \omega^*$. We denote by $C_\pi(X)$ the set of continuous function from X to the real numbers with the pointwise convergence topology, and $C_\pi(X, [0, 1])$ is the subspace of elements in $C_\pi(X)$ with values in the unit interval [0, 1]. It was proved in [ST1] that: (1) A generalized linearly ordered topological space (GLOTS) X is ultracompact if and only if X is pseudocompact; (2) $C_\pi(X, [0, 1])$ is ultrapseudocompact if and only if it is σ-pseudocompact; and (3) $C_\pi(X)$ is σ-ultrapseudocompact if and only if $C_\pi(X)$ is σ-pseudocompact. So, if $M, N \subset \omega^*$ we have: (i) For a GLOTS X, X is M-compact if and only if X is an N-pseudocompact space; (ii) $C_\pi(X, [0, 1])$ is M-pseudocompact if and only if it is σ-N-pseudocompact; and (iii) $C_\pi(X)$ is σ-M-pseudocompact if and only if $C_\pi(X)$ is σ-N-pseudocompact.

As it was pointed out in [ST1], there are spaces X for which $C_\pi(X, [0, 1])$ is ultrapseudocompact but not p-compact for any $p \in \omega^*$. So, p-compactness and p-pseudocompactness are not equivalent properties in the class of topological groups. On the other hand, for topological groups, p-pseudocompactness and $P_{RK}(p)$-pseudocompactness are equivalent. In fact, for topological groups, pseudocompactness and M-pseudocompactness, with $M \subset \omega^*$, are equivalent properties ([GFS]).

Example 5.12. There is a non-Frolík space which is $R(p)$-pseudocompact for every $p \in \omega^*$.

Proof. For each $p \in \omega^*$ and each open subset O of ω^*, $|R(p) \cap O| = 2^\omega$ (Actually, $|S(p) \cap O| = 2^\omega$). It follows immediately from the fact that $S(p)$ is dense and the fact that every open subset of ω^* has cellularity 2^ω. Let $\mathcal{B} = \{B_\lambda : \lambda < 2^\omega\}$ be a base of ω^*. Recursively choose for each $\lambda < 2^\omega$ and each $p \in \omega^*$ two different points a^p_λ, b^p_λ in $B_\lambda \cap R(p)$, in such a way that $a^p_\lambda \notin \{b^p_\gamma : \gamma < \lambda\}$ and $b^p_\lambda \notin \{a^p_\gamma : \gamma \leq \lambda\}$. Let $X = \omega \cup \{a^p_\lambda : \lambda < 2^\omega$, and $p \in \omega^*\}$ and $Y = \omega \cup \{b^p_\lambda : \lambda < 2^\omega$, and $p \in \omega^*\}$, both with its topology inherited by $\beta\omega$. By Theorem 5.6, X and Y are $R(p)$-pseudocompact for every $p \in \omega^*$, but $X \times Y$ is not pseudocompact because the sequence of open subsets $\{(n, n)\}$ does not have any accumulation point in $X \times Y$. So X and Y do not belong to the class of Frolík $\mathcal{F}$. □

Problem 5.13. Is there a non Frolík space which is $S(p)$-pseudocompact for every $p \in \omega^*$?

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